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The Compound Matrix Method for Ordinary Differential Systems

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A generalization of the compound matrix method is presented to deal with eigenvalue and boundary-value problems involving unstable systems of ordinary differential equations. Details are given for fourth- and sixth-order problems. Next it is shown that a simple equivalence relation exists between the compounds of the solution matrices of the eigenvalue problems and their adjoints, and how this relation can be exploited to simplify the calculation of the adjoint eigenfunctions is discussed. Using the Orr–Sommerfeld problem as an example, it is also shown how the techniques used in the derivation of certain auxiliary systems, which play a crucial role in the generalization of the compound matrix method, can also provide an alternate method for the direct computation of certain quantities defined in terms of the eigenfunction. Finally, there is a brief discussion of the relationship between the compound matrix method and the Riccati method. © 1985 Acedemic Press, Inc.

1. INTRODUCTION

In two earlier papers [1, 2], we introduced the compound matrix method for the numerical solution of mathematically unstable linear eigenvalue and boundary-value problems with separated boundary conditions. Since those developments were mainly motivated by work on the Orr–Sommerfeld equation, the focus of our previous studies has been on problems involving single differential equations of the form

$$\phi^{\mathrm{IV}} - a_1\phi''' - a_2\phi'' - a_3\phi' - a_4\phi = f, \tag{1.1}$$

where a_i $(i = 1, 2, 3, 4)$ and f are (complex) functions of x and $0 \leqslant x \leqslant 1$ (say), and ϕ is required to satisfy an equal number of boundary conditions at the endpoints. The procedure outlined in [1, 2] consists of two essential steps. First, rather than

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attempting to compute a set of linearly independent solutions of (1.1) which satisfy the boundary conditions at $x=0$ (say), we compute the minors of the corresponding solution matrix by a direct numerical integration of the so-called compound matrix equations which these minors satisfy. Next, we derive a second-order auxiliary differential equation the coefficients of which are some of the minors determined in the first step. The solution to the boundary-value problem is then obtained by integrating the auxiliary equation from $x=1$ to 0 subject to the boundary conditions at $x=1$.

Because of the effectiveness of the compound matrix method in treating problems involving single equations of the Orr–Sommerfeld type, it is desirable to generalize the method to deal with other unstable differential systems, typically of order four and six, which frequently arise in the study of hydrodynamic stability. In this connection, we note that the necessary generalization of the first step of the method for systems can readily be found in the work of Schwarz [3] who gave a general algorithm for the derivation of the compound matrix equation associated with an nth order (complex) linear system of the form

$$\boldsymbol{\phi}' = \mathbf{A}(x)\boldsymbol{\phi} + \mathbf{f}. \tag{1.2}$$

On the other hand, the method given in [1, 2] for the derivation of the auxiliary equation requires substantial generalization. In Section 2, therefore, we treat in detail the general technique for the derivation of certain auxiliary *systems* for fourth- and sixth-order problems of the form (1.2).

In Section 3, we consider eigenvalue problems. In particular, we call attention to a simple equivalence relation that exists between the compounds of the solution matrices of the eigenvalue problems and their adjoints. We also discuss how this relation can be exploited to simplify the calculation of the adjoint eigenfunctions. In Section 4, we use some of the ideas discussed in Section 2 to derive an auxiliary system associated with the Orr–Sommerfeld equation. This auxiliary system provides an alternate method for the computation of certain quantities such as the disturbance Reynolds stress and the energy distribution which are of interest in the study of hydrodynamic stability. Finally, in Section 5, we discuss briefly the relationship between the compound matrix method and the Riccati method.

2. Boundary-Value Problems

2.1. *Fourth-Order Systems*

Consider the linear inhomogeneous system

$$\boldsymbol{\phi}' = \mathbf{A}(x)\boldsymbol{\phi} + \mathbf{f}(x), \qquad 0 \leqslant x \leqslant 1, \tag{2.1}$$

where $\mathbf{A}(x) = [a_{ij}(x)]$ is a 4×4 matrix, $\mathbf{f}(x) = [f_j(x)]^{\mathrm{T}}$ and the solution

$\boldsymbol{\phi} = [\phi_j(x)]^{\mathrm{T}}$ are 4×1 column vectors. We shall also suppose that the boundary conditions at $x = 0$ and $x = 1$ are given by

$$\mathbf{P}\boldsymbol{\phi}(0) = \mathbf{p} \tag{2.2a}$$

and

$$\mathbf{Q}\boldsymbol{\phi}(1) = \mathbf{q}, \tag{2.2b}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are 2×4 matrices of rank 2 and $\mathbf{p}$ and $\mathbf{q}$ are 2×1 column vectors. By superposition, the solution to the two-point boundary-value problem can then be written in the form

$$\boldsymbol{\phi} = \mathbf{g} + \alpha\mathbf{u} + \beta\mathbf{v}, \tag{2.3}$$

where $\mathbf{g}$ is any solution of (2.1) which satisfies the initial condition (2.2a), while $\mathbf{u}$ and $\mathbf{v}$ are two linearly independent solutions of the homogeneous system

$$\boldsymbol{\phi}' = \mathbf{A}(\mathrm{x})\boldsymbol{\phi} \tag{2.4}$$

subject to the initial condition

$$\mathbf{P}\boldsymbol{\phi}(0) = \mathbf{0}. \tag{2.5}$$

The constants α and β are determined by requiring that $\boldsymbol{\phi}$ satisfies the boundary condition (2.2b) at $x = 1$.

Rather than attempting to compute $\mathbf{g}$, $\mathbf{u}$, and $\mathbf{v}$ explicitly as is the case with the other initial-value methods, the first step of the compound matrix method is based on considering certain minors of the 4×3 solution matrix $\boldsymbol{\Phi}_0 = [\mathbf{guv}]$ of the inhomogeneous system (2.1) and the 4×2 solution matrix $\boldsymbol{\Phi} = [\mathbf{uv}]$ of the corresponding homogeneous system. We note that the six 2×2 minors of $\boldsymbol{\Phi}$ are

$$y_{ij} = \begin{vmatrix} u_i & v_i \\ u_j & v_j \end{vmatrix}, \tag{2.6}$$

for $i = 1, 2, 3$ and $j = i + 1, \ldots, 4$. In terms of these 2×2 minors, the four 3×3 minors of $\boldsymbol{\Phi}_0$ can be written as

$$z_{ijk} = g_i y_{jk} + g_j y_{ki} + g_k y_{ij}, \tag{2.7}$$

where $i = 1, 2$, $j = i + 1, \ldots, 4$ and $k = j + 1, \ldots, 4$. For later purposes, we also note the quadratic identity

$$y_{12} y_{34} - y_{13} y_{24} + y_{14} y_{23} = 0, \tag{2.8}$$

which can readily be obtained from the Laplace expansion of the determinant

$$\begin{vmatrix} u_1 & v_1 & 0 & 0 \\ u_2 & v_2 & u_2 & v_2 \\ u_3 & v_3 & u_3 & v_3 \\ u_4 & v_4 & u_4 & v_4 \end{vmatrix} = 0. \tag{2.9}$$

If we now arrange the 2×2 minors of $\mathbf{\Phi}$ according to the lexicographical order of their indices to form the 6×1 vector $\mathbf{y} = [y_{12}, y_{13}, \ldots, y_{34}]^{\mathrm{T}}$, then $\mathbf{y}$ is called the second compound of $\mathbf{\Phi}$ which can also be denoted by $C_2[\mathbf{\Phi}]$. Similarly $\mathbf{z} = C_3[\mathbf{\Phi}_0] = [z_{123}, z_{124}, z_{134}, z_{234}]^{\mathrm{T}}$ is called the third compound of $\mathbf{\Phi}_0$. By a direct calculation or by using the algorithm of Schwarz [3, p. 205], it is easy to show that

$$\mathbf{y}' = \mathbf{B}(x)\,\mathbf{y}, \tag{2.10}$$

where

$$\mathbf{B}(x) = \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}. \tag{2.11}$$

Similarly, $\mathbf{z}$ must satisfy the equation

$$\mathbf{z}' = \mathbf{C}(x)\mathbf{z} + \mathbf{D}(x)\mathbf{f}, \tag{2.12}$$

where

$$\mathbf{C}(x) = \begin{bmatrix} a_{11}+a_{22}+a_{33} & a_{34} & -a_{24} & a_{14} \\ a_{43} & a_{11}+a_{22}+a_{44} & a_{23} & -a_{13} \\ -a_{42} & a_{32} & a_{11}+a_{33}+a_{44} & a_{12} \\ a_{41} & -a_{31} & a_{21} & a_{22}+a_{33}+a_{44} \end{bmatrix}, \tag{2.13}$$

and

$$\mathbf{D}(x) = \begin{bmatrix} y_{23} & -y_{13} & y_{12} & 0 \\ y_{24} & -y_{14} & 0 & y_{12} \\ y_{34} & 0 & -y_{14} & y_{13} \\ 0 & y_{34} & -y_{24} & y_{23} \end{bmatrix}. \tag{2.14}$$

Moreover, the initial conditions for **y** and **z** can easily be derived from (2.2a) and (2.5) using (2.6) and (2.7).

Suppose now that **y** and **z** have been computed by integrating (2.10) and (2.12) from $x=0$ to 1 subject to the appropriate initial conditions, the next step of the compound matrix method requires that the solution of the boundary-value problem (2.1), (2.2) be obtained from an auxiliary second-order system.

First, we note that for fixed i, j with $i \neq j$, (2.3) gives

$$\phi_i - g_i = \alpha u_i + \beta v_i \tag{2.15}$$

and

$$\phi_j - g_j = \alpha u_j + \beta v_j, \tag{2.16}$$

whence it follows that

$$y_{ij}\alpha = (\phi_i - g_i)v_j - (\phi_j - g_j)v_i \tag{2.17}$$

and

$$-y_{ij}\beta = (\phi_i - g_i)u_j - (\phi_j - g_j)u_i. \tag{2.18}$$

On substituting (2.17) and (2.18) into

$$\phi_i' - g_i' = \alpha u_i' + \beta v_i' \tag{2.19}$$

and simplifying, we have

$$y_{ij}\phi_i' = a_{i\mu}(y_{\mu j}\phi_i - y_{\mu i}\phi_j - z_{i\mu j}) + y_{ij}f_i, \tag{2.20}$$

where the summation is to be taken over μ. Moreover, on interchanging i and j in (2.16), and on noting that $y_{ij} = -y_{ji}$ and $z_{i\mu j} = -z_{j\mu i}$, we obtain

$$y_{ij}\phi_j' = a_{j\mu}(y_{\mu j}\phi_i - y_{\mu i}\phi_j - z_{i\mu j}) + y_{ij}f_j. \tag{2.21}$$

Equations (2.20) and (2.21) thus form a closed system which can be used for the determination of ϕ_i and ϕ_j by integrating backward from $x=1$ to 0.

To determine the initial condition for $\boldsymbol{\phi}$ at $x=1$, we rewrite (2.3) in matrix form

$$\begin{bmatrix} \phi_1 - g_1 & u_1 & v_1 \\ \phi_2 - g_2 & u_2 & v_2 \\ \phi_3 - g_3 & u_3 & v_3 \\ \phi_4 - g_4 & u_4 & v_4 \end{bmatrix} \begin{bmatrix} 1 \\ -\alpha \\ -\beta \end{bmatrix} = 0. \tag{2.22}$$

This is a linear homogeneous system with a nontrivial solution $[1, -\alpha, -\beta]^{\mathrm{T}}$.

Hence the determinant of any three rows of the coefficient matrix must vanish. This in turn implies that

$$\mathbf{D}(x)\,\boldsymbol{\phi}(x)=\mathbf{z}(x), \tag{2.23}$$

where $\mathbf{D}(x)$ is given by (2.14). Incidentally, we note that the derivation of (2.23) from (2.22) provides a simple algorithm for computing the matrix $\mathbf{D}(x)$ in (2.14). By using row reduction and the quadratic identity (2.8), it is easy to show that the augmented matrix associated with (2.23) is of rank 2, i.e.,

$$[\mathbf{D}\mid\mathbf{z}]\sim\left[\begin{array}{c|c}\mathbf{D}^* & \mathbf{z}^*\\ \hline \mathbf{0} & \mathbf{0}\end{array}\right]=\left[\begin{array}{cccc|c} y_{23} & -y_{13} & y_{12} & 0 & z_{123}\\ 0 & y_{34} & -y_{24} & y_{23} & z_{234}\\ \hline 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]. \tag{2.24}$$

Clearly then the boundary conditions (2.2b) together with (2.24) form a system of four linear equations of the form

$$\left[\begin{array}{c}\mathbf{D}^*(1)\\ \hline \mathbf{Q}\end{array}\right]\phi(1)=\left[\begin{array}{c}\mathbf{z}^*(1)\\ \hline \mathbf{q}\end{array}\right]. \tag{2.25}$$

Thus $\boldsymbol{\phi}(1)$ can be uniquely determined provided that

$$\det\left[\begin{array}{c}\mathbf{D}^*(1)\\ \hline \mathbf{Q}\end{array}\right]\neq 0. \tag{2.26}$$

We note that this condition must be satisfied if the boundary-value problem (2.1), (2.2) is to have a unique solution. This follows from the fact, which can be verified by a direct calculation, that (2.2b) is equivalent to the condition that $\det[\mathbf{Q}\boldsymbol{\Phi}(1)]\neq 0$, where $\boldsymbol{\Phi}$ is any solution matrix of the homogeneous system (2.4) which satisfies the homogeneous boundary conditions $\mathbf{P}\boldsymbol{\Phi}(0)=\mathbf{0}$.

Suppose now that ϕ_i and ϕ_j are computed by integrating (2.20), (2.21) from $x=1$ to 0 and that the remaining two components of $\boldsymbol{\phi}$ are obtained algebraically by solving $\mathbf{D}^*(x)\,\boldsymbol{\phi}(x)=\mathbf{z}^*(x)$, then it is necessary to show that $\boldsymbol{\phi}$ thus determined is the solution of the boundary-value problem (2.1), (2.2).

First, we consider (2.20) and (2.21). Using (2.23), we can rewrite (2.20) and (2.21), respectively, as

$$y_{ij}(\phi_i'-a_{i\mu}\phi_\mu-f_i)=0 \tag{2.27}$$

and

$$y_{ij}(\phi_j'-a_{j\mu}\phi_\mu-f_j)=0. \tag{2.28}$$

Second, differentiating (2.23) and noting that $\mathbf{D}' = \mathbf{CD} - \mathbf{DA}$, we obtain

$$\mathbf{D}(\boldsymbol{\phi}' - \mathbf{A}\boldsymbol{\phi} - \mathbf{f}) = \mathbf{0} \qquad \text{or} \qquad \mathbf{D}^*(\boldsymbol{\phi}' - \mathbf{A}\boldsymbol{\phi} - \mathbf{f}) = \mathbf{0}. \tag{2.29}$$

Equations (2.27)–(2.29) can clearly be combined to obtain

$$\mathbf{H}(\boldsymbol{\phi}' - \mathbf{A}\boldsymbol{\phi} - \mathbf{f}) = \mathbf{0}, \tag{2.30}$$

where $\mathbf{H}$ is nonsingular. Hence $\boldsymbol{\phi}$ is a solution of (2.1) and it satisfies the initial condition (2.25).

Finally, we note that the solution of the boundary-value problem (2.1), (2.2) must also satisfy the initial condition (2.25) at $x = 1$. Hence by uniqueness, if $\boldsymbol{\phi}$ is the solution of the initial-value problem (2.1) and (2.25) as is presently the case, then it is also a solution of the boundary-value problem (2.1), (2.2).

2.2. *Sixth-Order Systems*

Although we have restricted our discussion in Subsection 2.1 to fourth-order problems, the basic ideas are clearly quite general. In this section, we outline the corresponding results for problems involving sixth-order systems. These results, together with those presented in Subsection 2.1, are therefore directly applicable to a large class of unstable boundary-value problems which frequently arise in the study of hydrodynamic stability.

To avoid repetition in our present discussion, direct reference will be made to specific equations in the previous section, but it should then be understood that the definitions of the various quantities involved must be suitably modified to deal with sixth-order systems. For example, the coefficient matrix $\mathbf{A}(x)$ in (2.1) is now 6×6, and $\mathbf{f}(x)$ and $\boldsymbol{\phi}(x)$ are 6×1 column vectors. Similarly in (2.2), $\mathbf{P}$ and $\mathbf{Q}$ are 3×6 matrices of rank 3, and $\mathbf{p}$ and $\mathbf{q}$ are 3×1 column vectors. Analogous to (2.3), we write

$$\boldsymbol{\phi} = \mathbf{g} + \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}, \tag{2.31}$$

where $\mathbf{g}$ is any solution of (2.1) which satisfies the initial condition (2.2a), while $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three linearly independent solutions of (2.4), (2.5).

Consider now the solution matrices $\boldsymbol{\Phi}_0 = [\mathbf{guvw}]$ and $\boldsymbol{\Phi} = [\mathbf{uvw}]$. The twenty 3×3 minors of $\boldsymbol{\Phi}$ are

$$y_{ijk} = \begin{vmatrix} u_i & v_i & w_i \\ u_j & v_j & w_j \\ u_k & v_k & w_k \end{vmatrix}, \tag{2.32}$$

where $i = 1, 2, 3, 4$, $j = i + 1, \ldots, 5$, and $k = j + 1, \ldots, 6$. The fifteen 4×4 minors of $\boldsymbol{\Phi}_0$ can be written as

$$z_{ijkl} = g_i\, y_{jkl} - g_j\, y_{kli} + g_k\, y_{lij} - g_l\, y_{ijk}, \tag{2.33}$$

where $i=1,2,\ldots,6$, $j=i+1,\ldots,6$, $k=j+1,\ldots,6$, and $l=k+1,\ldots,6$. Moreover, by considering the Laplace expansion by complementary minors of the determinant

$$\begin{vmatrix} u_i & v_i & w_i & 0 & 0 & 0 \\ u_j & v_j & w_j & u_j & v_j & w_j \\ u_m & v_m & w_m & u_m & v_m & w_m \\ u_k & v_k & w_k & u_k & v_k & w_k \\ u_l & v_l & w_l & u_l & v_l & w_l \\ u_m & v_m & w_m & 0 & 0 & 0 \end{vmatrix} = 0, \tag{2.34}$$

we obtain 30 quadratic identities of the form

$$y_{ijm}\,y_{klm} - y_{ikm}\,y_{jlm} + y_{ilm}\,y_{jkm} = 0, \tag{2.35}$$

where $m=1,\ldots,6$; $i, j, k, l \neq m$; $1 \leqslant i < j < k < l \leqslant 6$.

If we let $\mathbf{y} = C_3[\mathbf{\Phi}] = [y_{123}, y_{124},\ldots, y_{456}]^{\mathrm{T}}$ and $\mathbf{z} = C_4[\mathbf{\Phi}_0] = [z_{1234}, z_{1235},\ldots, z_{3456}]^{\mathrm{T}}$, then $\mathbf{y}$ and $\mathbf{z}$ satisfy the compound matrix equations (2.10) and (2.12), respectively. The elements of the matrices $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ in these equations are given, respectively, in Tables I, II, and III.

Corresponding to (2.20) and (2.21), we now have the auxiliary system

$$y_{ijk}\phi_i' = a_{i\mu}(y_{\mu jk}\phi_i - y_{\mu ik}\phi_j + y_{\mu ij}\phi_k - z_{i\mu jk}) + y_{ijk} f_i, \tag{2.36}$$

$$y_{ijk}\phi_j' = a_{j\mu}(y_{\mu jk}\phi_i - y_{\mu ik}\phi_j + y_{\mu ij}\phi_k - z_{i\mu jk}) + y_{ijk} f_j, \tag{2.37}$$

$$y_{ijk}\phi_k' = a_{k\mu}(y_{\mu jk}\phi_i - y_{\mu ik}\phi_j + y_{\mu ij}\phi_k - z_{i\mu jk}) + y_{ijk} f_k. \tag{2.38}$$

Furthermore, by using the quadratic identities (2.35) and after a somewhat lengthy calculation, we can show that the augmented matrix $[\mathbf{D}(x) \mid \mathbf{z}(x)]$ is of rank 3, i.e.,

$$[\mathbf{D} \mid \mathbf{z}] \sim \left[\begin{array}{c|c} \mathbf{D}^* & \mathbf{z}^* \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cccccc|c} y_{234} & -y_{134} & y_{124} & -y_{123} & 0 & 0 & z_{1234} \\ 0 & y_{345} & -y_{245} & y_{235} & -y_{234} & 0 & z_{2345} \\ 0 & 0 & y_{456} & -y_{356} & y_{346} & -y_{345} & z_{3456} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.39}$$

Clearly then the initial condition for $\boldsymbol{\phi}$ at $x=1$ can be uniquely determined using (2.25). Thus once $\mathbf{y}$ and $\mathbf{z}$ have been computed by integrating (2.10)–(2.12) from $x=0$ to 1, the solution $\boldsymbol{\phi}$ can be obtained by integrating (2.36)–(2.38) backward from $x=1$ to 0. Moreover, as in the fourth-order case, we can show that $\boldsymbol{\phi}$ is indeed the solution of the sixth-order boundary-value problem (2.1), (2.2).

TABLE I

The Matrix **B** Given in Terms of the Elements a_{ij} for Sixth-Order Problems[a]

s_{123}	a_{34}	a_{35}	a_{36}	$-a_{24}$	$-a_{25}$	$-a_{26}$	0	0	0	a_{14}	a_{15}	a_{16}	0	0	0	0	0	0	0
a_{43}	s_{124}	a_{45}	a_{46}	a_{23}	0	0	$-a_{25}$	$-a_{26}$	0	$-a_{13}$	0	0	a_{15}	a_{16}	0	0	0	0	0
a_{53}	a_{54}	s_{125}	a_{56}	0	a_{23}	0	a_{24}	0	$-a_{26}$	0	$-a_{13}$	0	$-a_{14}$	0	a_{16}	0	0	0	0
a_{63}	a_{64}	a_{65}	s_{126}	0	0	a_{23}	0	a_{24}	a_{25}	0	0	$-a_{13}$	0	$-a_{14}$	$-a_{15}$	0	0	0	0
$-a_{42}$	a_{32}	0	0	s_{134}	a_{45}	a_{46}	$-a_{35}$	$-a_{36}$	0	a_{12}	0	0	0	0	0	a_{15}	a_{16}	0	0
$-a_{52}$	0	a_{32}	0	a_{54}	s_{135}	a_{56}	a_{34}	0	$-a_{36}$	0	a_{12}	0	0	0	0	$-a_{14}$	0	a_{16}	0
$-a_{62}$	0	0	a_{32}	a_{64}	a_{65}	s_{136}	0	a_{34}	a_{35}	0	0	a_{12}	0	0	0	0	$-a_{14}$	$-a_{15}$	0
0	$-a_{52}$	a_{42}	0	$-a_{53}$	a_{43}	0	s_{145}	a_{56}	$-a_{46}$	0	0	0	a_{12}	0	0	a_{13}	0	0	a_{16}
0	$-a_{62}$	0	a_{42}	$-a_{63}$	0	a_{43}	a_{65}	s_{146}	a_{45}	0	0	0	0	a_{12}	0	0	a_{13}	0	$-a_{15}$
0	0	$-a_{62}$	a_{52}	0	$-a_{63}$	a_{53}	$-a_{64}$	a_{54}	s_{156}	0	0	0	0	0	a_{12}	0	0	a_{13}	a_{14}
a_{41}	$-a_{31}$	0	0	a_{21}	0	0	0	0	0	a_{234}	a_{45}	a_{46}	$-a_{35}$	$-a_{36}$	0	a_{25}	a_{26}	0	0
a_{51}	0	$-a_{31}$	0	0	a_{21}	0	0	0	0	a_{54}	s_{235}	a_{56}	a_{34}	0	$-a_{36}$	$-a_{24}$	0	a_{26}	0
a_{61}	0	0	$-a_{31}$	0	0	a_{21}	0	0	0	a_{64}	a_{65}	s_{236}	0	a_{34}	a_{35}	0	$-a_{24}$	$-a_{25}$	0
0	a_{51}	$-a_{41}$	0	0	0	0	a_{21}	0	0	$-a_{53}$	a_{43}	0	s_{245}	a_{56}	$-a_{46}$	a_{23}	0	0	a_{26}
0	a_{61}	0	$-a_{41}$	0	0	0	0	a_{21}	0	$-a_{63}$	0	a_{43}	a_{65}	s_{246}	a_{45}	0	a_{23}	0	$-a_{25}$
0	0	a_{61}	$-a_{51}$	0	0	0	0	0	a_{21}	0	$-a_{63}$	a_{53}	$-a_{64}$	a_{54}	s_{256}	0	0	a_{23}	a_{24}
0	0	0	0	a_{51}	$-a_{41}$	0	a_{31}	0	0	a_{52}	$-a_{42}$	0	a_{32}	0	0	s_{345}	a_{56}	$-a_{46}$	a_{36}
0	0	0	0	a_{61}	0	$-a_{41}$	0	a_{31}	0	a_{62}	0	$-a_{42}$	0	a_{32}	0	a_{65}	s_{346}	a_{45}	$-a_{35}$
0	0	0	0	0	a_{61}	$-a_{51}$	0	0	a_{31}	0	a_{62}	$-a_{52}$	0	0	a_{32}	$-a_{64}$	a_{54}	s_{356}	a_{34}
0	0	0	0	0	0	0	a_{61}	$-a_{51}$	a_{41}	0	0	0	a_{62}	$-a_{52}$	a_{42}	a_{63}	$-a_{53}$	a_{43}	s_{456}

[a] s_{ilm} denotes the sum of the diagonal elements of the ith, lth, and mth row of A.

TABLE II

The Matrix C Given in Terms of the Elements a_{ij} for Sixth-Order Problems[a]

s_{1234}	a_{45}	a_{46}	$-a_{35}$	$-a_{36}$	0	a_{25}	a_{26}	0	0	$-a_{15}$	$-a_{16}$	0	0	0
a_{54}	s_{1235}	a_{56}	a_{34}	0	$-a_{36}$	$-a_{24}$	0	a_{26}	0	a_{14}	0	$-a_{16}$	0	0
a_{64}	a_{65}	s_{1236}	0	a_{34}	a_{35}	0	$-a_{24}$	$-a_{25}$	0	0	a_{14}	a_{15}	0	0
$-a_{53}$	a_{43}	0	s_{1245}	a_{56}	$-a_{46}$	a_{23}	0	0	a_{26}	$-a_{13}$	0	0	$-a_{16}$	0
$-a_{63}$	0	a_{43}	a_{65}	s_{1246}	a_{45}	0	a_{23}	0	$-a_{25}$	0	$-a_{13}$	0	a_{15}	0
0	$-a_{63}$	a_{53}	$-a_{64}$	a_{54}	s_{1256}	0	0	a_{23}	a_{24}	0	0	$-a_{13}$	$-a_{14}$	0
a_{52}	$-a_{42}$	0	a_{32}	0	0	s_{1345}	a_{56}	$-a_{46}$	a_{36}	a_{12}	0	0	0	$-a_{16}$
a_{62}	0	$-a_{42}$	0	a_{32}	0	a_{65}	s_{1346}	a_{45}	$-a_{35}$	0	a_{12}	0	0	a_{15}
0	a_{62}	$-a_{52}$	0	0	a_{32}	$-a_{64}$	a_{54}	s_{1356}	a_{34}	0	0	a_{12}	0	$-a_{14}$
0	0	0	a_{62}	$-a_{52}$	a_{42}	a_{63}	$-a_{53}$	a_{43}	s_{1456}	0	0	0	a_{12}	a_{13}
$-a_{51}$	a_{41}	0	$-a_{31}$	0	0	a_{21}	0	0	0	s_{2345}	a_{56}	$-a_{46}$	a_{36}	$-a_{26}$
$-a_{61}$	0	a_{41}	0	$-a_{31}$	0	0	a_{21}	0	0	a_{65}	s_{2346}	a_{45}	$-a_{35}$	a_{25}
0	$-a_{61}$	a_{51}	0	0	$-a_{31}$	0	0	a_{21}	0	$-a_{64}$	a_{54}	s_{2356}	a_{34}	$-a_{24}$
0	0	0	$-a_{61}$	a_{51}	$-a_{41}$	0	0	0	a_{21}	a_{63}	$-a_{53}$	a_{43}	s_{2456}	a_{23}
0	0	0	0	0	0	$-a_{61}$	a_{51}	$-a_{41}$	a_{31}	$-a_{62}$	a_{52}	$-a_{42}$	a_{32}	s_{3456}

[a] s_{ilmn} denotes the sum of the diagonal elements of the ith, lth, mth, and nth row of $\mathbf{A}$.

3. Eigenvalue Problems and Their Adjoints

Consider the linear eigenvalue problem defined by

$$\boldsymbol{\phi}' = \mathbf{A}(x, \lambda)\boldsymbol{\phi}, \qquad 0 \leqslant x \leqslant 1, \tag{3.1}$$

$$\mathbf{P}\boldsymbol{\phi}(0) = \mathbf{0} \tag{3.2a}$$

and

$$\mathbf{Q}\boldsymbol{\phi}(1) = \mathbf{0}, \tag{3.2b}$$

where $\mathbf{A}(x, \lambda)$ is a $2r \times 2r$ matrix, $\mathbf{P}$ and $\mathbf{Q}$ are $r \times 2r$ matrices of rank r, $\boldsymbol{\phi}$ is a $2r \times 1$

TABLE III

The Matrix **D** Given in Terms of the Elements y_{ijk} for Sixth-Order Problems

y_{234}	$-y_{134}$	y_{124}	$-y_{123}$	0	0
y_{235}	$-y_{135}$	y_{125}	0	$-y_{123}$	0
y_{236}	$-y_{136}$	y_{126}	0	0	$-y_{123}$
y_{245}	$-y_{145}$	0	y_{125}	$-y_{124}$	0
y_{246}	$-y_{146}$	0	y_{126}	0	$-y_{124}$
y_{256}	$-y_{156}$	0	0	y_{126}	$-y_{125}$
y_{345}	0	$-y_{145}$	y_{135}	$-y_{134}$	0
y_{346}	0	$-y_{146}$	y_{136}	0	$-y_{134}$
y_{356}	0	$-y_{156}$	0	y_{136}	$-y_{135}$
y_{456}	0	0	$-y_{156}$	y_{146}	$-y_{145}$
0	y_{345}	$-y_{245}$	y_{235}	$-y_{234}$	0
0	y_{346}	$-y_{246}$	y_{236}	0	$-y_{234}$
0	y_{356}	$-y_{256}$	0	y_{236}	$-y_{235}$
0	y_{456}	0	$-y_{256}$	y_{246}	$-y_{245}$
0	0	y_{456}	$-y_{356}$	y_{346}	$-y_{345}$

vector and λ is the eigenvalue parameter. Depending on whether $r=2$ or 3, Eqs. (3.1), (3.2) define a fourth- or sixth-order problem for which many of the results given in Section 2 are directly applicable.

Consistent with the notation of Section 2, we let $\mathbf{y}$ be the rth compound of the $2r \times r$ solution matrix $\mathbf{\Phi}$ of (3.1) which satisfies the boundary condition (3.2a) at $x=0$. Clearly then $\mathbf{y}$ must satisfy the compound matrix equation [cf. (2.10)]

$$\mathbf{y}' = \mathbf{B}(x, \lambda)\mathbf{y}, \tag{3.3}$$

where the matrix $\mathbf{B}(x, \lambda)$ can be derived from $\mathbf{A}(x, \lambda)$ as discussed in Section 2. By

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using the rule that the compound of the product of two matrices is equal to the product of their compounds, it then follows from (3.2b) that

$$C_r[\mathbf{Q}\mathbf{\Phi}(1)] = C_r[\mathbf{Q}] \cdot C_r[\mathbf{\Phi}(1)] = 0. \tag{3.4}$$

Equation (3.4) is the appropriate eigenvalue relation for (3.1), (3.2) and it is equivalent to requiring that a certain linear combination of the elements of $\mathbf{y}$ vanish at $x = 1$. Thus to determine the eigenvalue, we can repeatedly integrate the compound matrix equation (3.3) from $x = 0$ to 1, while a Newton-type iteration scheme is used to vary λ until the eigenvalue relation (3.4) is satisfied.

Next we note that $\boldsymbol{\phi}$ must also satisfy the auxiliary systems (2.20), (2.21) or (2.36)–(2.38) with $\mathbf{z} \equiv 0$ and $\mathbf{f} \equiv 0$. The initial condition for $\boldsymbol{\phi}$ at $x = 1$ can be obtained in the same manner as discussed in Section 2. Thus, once we have computed the eigenvalue, we can obtain $\boldsymbol{\phi}$ by integrating (2.20), (2.21) or (2.36)–(2.38) from $x = 1$ to 0. Moreover, using an argument identical to the one used for boundary-value problems, we can show that $\boldsymbol{\phi}$ is indeed an eigenfunction of (3.1), (3.2).

Now, consider the adjoint problem of (3.1), (3.2) consisting of the system

$$\boldsymbol{\phi}^{\dagger\prime} = -\mathbf{A}^H(x, \lambda)\boldsymbol{\phi}^\dagger, \tag{3.5}$$

and the adjoint boundary conditions

$$\mathbf{\Phi}^H(0)\boldsymbol{\phi}^\dagger(0) = 0 \tag{3.6a}$$

and

$$\mathbf{\Phi}^H(1)\boldsymbol{\phi}^\dagger(1) = 0, \tag{3.6b}$$

where the superscript H denotes the conjugate transpose, and $\mathbf{\Phi}$ is the solution matrix of (3.1) which satisfies the boundary condition (3.2a). To obtain the adjoint eigenfunction $\boldsymbol{\phi}^\dagger$, we may, of course, follow the same procedure for solving (3.1), (3.2) and first integrate the compound matrix system associated with (3.5), i.e.,

$$\mathbf{y}^{\dagger\prime} = -\mathbf{B}^H(x, \lambda)\mathbf{y}^\dagger, \tag{3.7}$$

from $x = 0$ to 1. Here $\mathbf{y}^\dagger$ denotes the rth compound of the $2r \times r$ solution matrix $\mathbf{\Phi}^\dagger$ of (3.5) which satisfies the boundary condition (3.6a). Once we have computed $\mathbf{y}^\dagger$ for $0 \leqslant x \leqslant 1$, the adjoint eigenfunction $\boldsymbol{\phi}^\dagger$ can be obtained by integrating the appropriate auxiliary system (either (2.20), (2.21) or (2.36)–(2.38)) from $x = 1$ to 0, except that the elements of $\mathbf{y}$ in these systems must now be replaced by the corresponding elements of $\mathbf{y}^\dagger$. On the other hand, the above procedure can be simplified and the need for integrating (3.7) can be circumvented on noting that a simple equivalence relation exists between the elements of $\mathbf{y}$ and $\mathbf{y}^\dagger$.

For the purpose of the present discussion, we first assume that the matrix $\mathbf{A}$ is in *normal* form, i.e., the diagonal elements of **A** are all zero. It then follows that $b_{ii} = 0$. By a straightforward but somewhat tedious calculation, it can be verified that

$$\mathbf{y}^{\dagger} = \mathbf{T}\mathbf{y}^{*}, \tag{3.8}$$

where $\mathbf{y}^*$ is the complex conjugate of $\mathbf{y}$, and **T** is a constant *antidiagonal* matrix of the form

$$\mathbf{T} = \begin{bmatrix} 0 & & t_{1n} \\ & \cdot^{\cdot^{\cdot}} & \\ t_{n1} & & 0 \end{bmatrix}. \tag{3.9}$$

For fourth-order ($r = 2$) problems, **T** is 6×6 and

$$\text{antidiag}\, \mathbf{T} = [t_{16}, \ldots, t_{61}] = [1, -1, 1, 1, -1, 1]. \tag{3.10}$$

Similarly for sixth-order ($r = 3$) problems, **T** is 20×20 and

$$\text{antidiag}\, \mathbf{T}$$
$$= [1, -1, 1, -1, 1, -1, 1, 1, -1, 1, -1, 1, -1, -1, 1, -1, 1, -1, 1, -1]. \tag{3.11}$$

Now, consider the case where the coefficient matrix **A** is not in normal form. We then let $\mathbf{N} = \mathbf{A} - \mathbf{\Lambda}$, where $\mathbf{\Lambda} = \text{diag}\, \mathbf{A}$ and

$$\boldsymbol{\phi} = \mathbf{E}\hat{\boldsymbol{\phi}} \qquad \text{where} \quad \mathbf{E} = \mathbf{E}(x) = \exp\left[\int_0^x \mathbf{\Lambda}\, dx\right]. \tag{3.12}$$

Then (3.1) becomes

$$\hat{\boldsymbol{\phi}}' = (\mathbf{E}^{-1}\mathbf{N}\mathbf{E})\hat{\boldsymbol{\phi}}, \tag{3.13}$$

where $\mathbf{E}^{-1}\mathbf{N}\mathbf{E}$ is a matrix whose diagonal elements are zero. Corresponding to (3.2), we have

$$\mathbf{P}\hat{\boldsymbol{\phi}}(0) = \mathbf{0} \qquad \text{and} \qquad [\mathbf{Q}\mathbf{E}(1)]\, \hat{\boldsymbol{\phi}}(1) = \mathbf{0}. \tag{3.14}$$

It is easy to show that $\boldsymbol{\phi}^{\dagger} = (\mathbf{E}^*)^{-1}\hat{\boldsymbol{\phi}}^{\dagger}$, where $\mathbf{E}^*$ is the complex conjugate of **E**, and $\hat{\boldsymbol{\phi}}^{\dagger}$ is the adjoint eigenfunction of (3.13), (3.14). Thus if we let $\mathbf{\Phi}$ and $\hat{\mathbf{\Phi}}$ be, respectively, the solution matrices of (3.1) and (3.13) which satisfy the boundary conditions (3.2a) and (3.14a), and $\mathbf{\Phi}^{\dagger}$ and $\hat{\mathbf{\Phi}}^{\dagger}$ be the solution matrices of the corresponding adjoint problems, then

$$\mathbf{\Phi} = \mathbf{E}\hat{\mathbf{\Phi}} \qquad \text{and} \qquad \mathbf{\Phi}^{\dagger} = (\mathbf{E}^*)^{-1}\hat{\mathbf{\Phi}}^{\dagger}. \tag{3.15}$$

By taking the rth compounds of Eqs. (3.15), we obtain

$$\mathbf{y} = C_r(\mathbf{E})\hat{\mathbf{y}} \qquad \text{and} \qquad \mathbf{y}^{\dagger} = C_r[(\mathbf{E}^*)^{-1}]\hat{\mathbf{y}}^{\dagger}, \tag{3.16}$$

where $\hat{\mathbf{y}} = C_r(\hat{\mathbf{\Phi}})$ and $\hat{\mathbf{y}}^\dagger = C_r(\hat{\mathbf{\Phi}}^\dagger)$. Since $\hat{\mathbf{y}}^\dagger = \mathbf{T}\hat{\mathbf{y}}^*$, it follows from (3.16) that

$$\mathbf{y}^\dagger = C_r[(\mathbf{E}^*)^{-1}]\,\mathbf{T}C_r[(\mathbf{E}^*)^{-1}]\,\mathbf{y}^* = [w(x)]^{-1}\mathbf{T}\mathbf{y}^*, \tag{3.17}$$

where

$$w(x) = \exp\left[\int_0^x \operatorname{tr}(\mathbf{A}^H)\,dx\right]. \tag{3.18}$$

Thus, once we have obtained $\mathbf{y}$ by integrating (3.3), $\mathbf{y}^\dagger$ can be obtained from (3.10), (3.11) and (3.17), (3.18) with little further computation. Moreover, when these results are applied to the actual computation of the adjoint eigenfunction, it is not necessary to evaluate $w(x)$. This is due to the fact that in replacing the elements of $\mathbf{y}$ in the auxiliary systems (2.20), (2.21) or (2.36)–(2.38) by the appropriate elements of $\mathbf{y}^\dagger$, an overall multiplicative factor of $\mathbf{y}^\dagger$ (i.e., $w(x)$) can clearly be omitted.

We note that in most weakly nonlinear theories of hydrodynamic stability, it is often necessary to solve first an eigenvalue problem which governs the linear stability, and then a sequence of inhomogeneous boundary-value problems. For unique solutions to exist for these problems, it is necessary to require that their inhomogeneous terms be orthogonal to the adjoint eigenfunction of the linear stability theory. This in turn provides the various conditions needed for the systematic determination of the Landau constants in the Landau amplitude equation. An application of the procedure outlined in this section for computing the adjoint eigenfunction can therefore lead to a simplification of some aspects of nonlinear stability calculations.

4. A Related Application

In this section, we wish to show, by means of the Orr–Sommerfeld problem, how certain quantities defined in terms of the eigenfunction of an eigenvalue problem can be computed directly using an appropriate auxiliary system of equations.

In system form, the Orr–Sommerfeld equation, which governs the linear stability of parallel shear flows, is given by

$$\boldsymbol{\phi}' = \mathbf{A}\boldsymbol{\phi}, \tag{4.1}$$

where

$$\boldsymbol{\phi} = [\phi, \phi', \phi'', \phi''']^T, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{41} & 0 & a_{43} & 0 \end{bmatrix}, \tag{4.2}$$

with $a_{41} = -\{\alpha^4 + i\alpha R[\alpha^2(U-c) + U'']\}$ and $a_{43} = 2\alpha^2 + i\alpha R(U-c)$. Here ϕ is the amplitude of the disturbance stream function, $U(x)$ is the basic velocity distribution, α and R are real parameters and c is the (possibly complex) eigenvalue parameter. For plane Poiseuille flow on the interval $0 \leqslant x \leqslant 2$, we have $U(x) = x(2-x)$. If we consider only symmetric modes, then the problem can be studied on the interval $0 \leqslant x \leqslant 1$ with boundary conditions

$$\phi(0) = \phi'(0) = 0 \tag{4.3a}$$

and

$$\phi'(1) = \phi'''(1) = 0. \tag{4.3b}$$

The solution of (4.1)–(4.3) using the compound matrix method has already been discussed in [1]. For our purpose, we need only note that the eigenfunction ϕ satisfies the second-order auxiliary equation (cf. [1, Eq. (12)])

$$y_1\phi'' - y_2\phi' + y_4\phi = 0, \tag{4.4}$$

where y_1, y_2, and y_4 are elements of the second compound $\mathbf{y}$ of the 4×2 solution matrix of (4.1) which satisfies the homogeneous boundary condition at $x = 0$. In the notation of Section 2, $y_1 = y_{12}$, $y_2 = y_{13}$ and $y_4 = y_{23}$. It is also convenient to rewrite ϕ, y_2/y_1 and y_4/y_1 in terms of their real and imaginary parts, i.e.,

$$\phi = \phi_r + i\phi_i, \qquad y_2/y_1 = s_r + is_i, \qquad y_4/y_1 = t_r + it_i. \tag{4.5}$$

Equation (4.4) then becomes

$$\begin{bmatrix} \phi_r \\ \phi_i \\ \phi_r' \\ \phi_i' \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -t_r & t_i & s_r & -s_i \\ -t_i & -t_r & s_i & s_r \end{bmatrix} \begin{bmatrix} \phi_r \\ \phi_i \\ \phi_r' \\ \phi_i' \end{bmatrix}. \tag{4.6}$$

Consider now the so-called Reynolds stress function $\phi_r\phi_i' - \phi_r'\phi_i$. Differentiating and then using (4.6), we have

$$(\phi_r\phi_i' - \phi_r'\phi_i)' = s_r(\phi_r\phi_i' - \phi_r'\phi_i) + s_i(\phi_r\phi_r' + \phi_i\phi_i') - t_i(\phi_r^2 + \phi_i^2). \tag{4.7}$$

If we now define

$$\begin{aligned} \tau_1 &= \phi_r\phi_i' - \phi_r'\phi_i, \qquad & \tau_2 &= \phi_r\phi_r' + \phi_i'\phi_i, \\ \tau_3 &= \phi_r^2 + \phi_i^2, & \tau_4 &= \phi_r'^2 + \phi_i'^2, \end{aligned} \tag{4.8}$$

then it is easy to show that

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix}' = \begin{bmatrix} s_r & s_i & -t_i & 0 \\ -s_i & s_r & -t_r & 1 \\ 0 & 2 & 0 & 0 \\ -2t_i & -2t_r & 0 & 2s_r \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix}. \tag{4.9}$$

Moreover, if we fix the normalization so that $\phi(1) = 1$ (say), the boundary conditions (4.3b) give

$$\tau_1(1) = \tau_2(1) = \tau_4(1) = 0 \quad \text{and} \quad \tau_3(1) = 1. \tag{4.10}$$

Equations (4.9), (4.10) thus show that once we have determined the eigenvalue c and obtained $\mathbf{y}$ by integrating the appropriate compound matrix system which $\mathbf{y}$ satisfies, then $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3, \tau_4]^T$ can be obtained directly by integrating (4.9) from $x = 1$ to 0 without having first to compute ϕ.

To test the effectiveness of this approach, we have computed $\boldsymbol{\tau}$ using (4.9) for the eigenmode corresponding to

$$\alpha = 1, \qquad R = 10^5, \qquad \text{and} \qquad c = 0.2375 + 0.0037i. \tag{4.11}$$

The results we obtained for the Reynolds stress distribution τ_1 were found to be in excellent agreement with previous results (see, e.g., [4, p. 223]). Our results for τ_1 and τ_3, which can be interpreted as the energy distribution of the disturbance, are shown in Fig. 1.

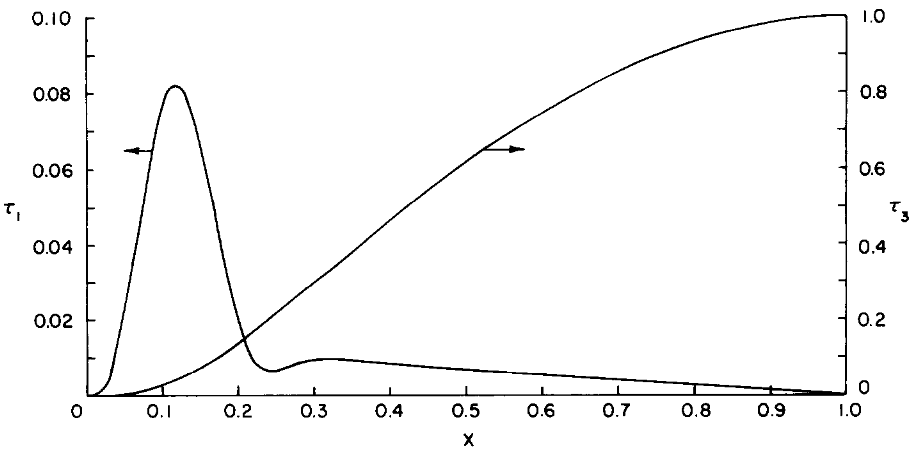

FIG. 1. The disturbance Reynolds stress τ_1 and the energy distribution τ_3 for plane Poiseuille flow at $\alpha = 1$, $R = 10^5$ and $c = 0.2375 + 0.0037i$.

5. Relationship to the Riccati Method

In [2], we discussed the relationship between the Riccati method and the compound matrix method for fourth-order boundary-value problems. It is of interest to consider briefly a similar result for sixth-order problems. In particular, we wish to record the relationship between the Riccati matrix and the various components of the compound matrix $\mathbf{y}$.

For simplicity, we consider an eigenvalue problem consisting of a sixth-order homogeneous system of the form

$$\begin{aligned} \mathbf{u}' &= \mathbf{A}_{11}(x)\mathbf{u} + \mathbf{A}_{12}(x)\mathbf{v}, \\ \mathbf{v}' &= \mathbf{A}_{21}(x)\mathbf{u} + \mathbf{A}_{22}(x)\mathbf{v}. \end{aligned} \tag{5.1}$$

Here $\mathbf{A}_{11}$, $\mathbf{A}_{12}, \mathbf{A}_{21}$, and $\mathbf{A}_{22}$ are 3×3 matrices, $\mathbf{u} = [\phi_1, \phi_2, \phi_3]^{\mathrm{T}}$ and $\mathbf{v} = [\phi_4, \phi_5, \phi_6]^{\mathrm{T}}$ for $0 \leqslant x \leqslant 1$. We shall also suppose that the boundary conditions at $x=0$ are given by $\mathbf{u}(0)=\mathbf{0}$. The exact form of the homogeneous boundary conditions at $x=1$ need not, however, concern us. The first step in the application of the Riccati method is to define a transformation of the form

$$\mathbf{u} = \mathbf{R}\mathbf{v}. \tag{5.2}$$

The 3×3 Riccati matrix $\mathbf{R}$ must then satisfy the equation [5],

$$\mathbf{R}' = \mathbf{A}_{11}\mathbf{R} - \mathbf{R}\mathbf{A}_{22} - \mathbf{R}\mathbf{A}_{21}\mathbf{R} + \mathbf{A}_{12}, \tag{5.3}$$

and it can easily be seen from (5.2) that the initial condition for $\mathbf{R}$ at $x=0$ is $\mathbf{R}(0)=\mathbf{0}$. To obtain the relationship between the elements of $\mathbf{R}$ and the components of $\mathbf{y}$, we consider the general form of (5.2), i.e.,

$$\mathbf{U} = \mathbf{R}\mathbf{V} \qquad \text{or} \qquad \mathbf{R} = \mathbf{U}\mathbf{V}^{-1}, \tag{5.4}$$

where

$$\mathbf{U} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{V} = \begin{bmatrix} u_4 & v_4 & w_4 \\ u_5 & v_5 & w_5 \\ u_6 & v_6 & w_6 \end{bmatrix}. \tag{5.5}$$

A short calculation then gives

$$\mathbf{V}^{-1} = \frac{1}{\det \mathbf{V}} \begin{bmatrix} v_5 w_6 - v_6 w_5 & -(v_4 w_6 - v_6 w_4) & v_4 w_5 - v_5 w_4 \\ -(u_5 w_6 - u_6 w_5) & u_4 w_6 - u_6 w_4 & -(u_4 w_5 - u_5 w_4) \\ u_5 v_6 - u_6 v_5 & -(u_4 v_6 - u_6 v_4) & u_4 v_5 - u_5 v_4 \end{bmatrix}. \tag{5.6}$$

It follows from (5.4)–(5.6) and (2.32) that

$$\mathbf{R}=\frac{1}{y_{456}}\begin{bmatrix} y_{156} & -y_{146} & y_{145} \\ y_{256} & -y_{246} & y_{245} \\ y_{356} & -y_{346} & y_{345} \end{bmatrix}. \tag{5.7}$$

Similarly, it can easily be verified that when $\mathbf{R}$ is nonsingular, the inverse Riccati matrix $\mathbf{S}$ is given by

$$\mathbf{S}=\mathbf{R}^{-1}=\frac{1}{y_{123}}\begin{bmatrix} y_{234} & -y_{134} & y_{124} \\ y_{235} & -y_{135} & y_{125} \\ y_{236} & -y_{136} & y_{126} \end{bmatrix}. \tag{5.8}$$

Equations (5.7) and (5.8) can be compared to the corresponding results given by Davey [6] for fourth-order problems.

6. Discussion

The major aim of the present paper is to provide a generalization of the compound matrix method to deal with eigenvalue and boundary-value problems involving unstable ordinary differential systems. In contrast with most other initial-value techniques in which the solution must be computed as a linear combination of a set of basis solutions, the present method first derives a lower order auxiliary system whose solutions automatically satisfy the boundary conditions at one of the two endpoints. The solution to the original problem is then obtained by integrating the auxiliary system subject to the boundary conditions at the second endpoint. The application of this technique thus has the effect of "pinning down" the solution of the boundary-value problem at *both* endpoints. In this regard the compound matrix method is perhaps quite similar to other boundary-value techniques such as the finite difference method and indeed this is the chief reason for the effectiveness of the method in treating highly unstable problems.

We also believe that with some refinements, the present technique should also be very effective in dealing with problems of the singular perturbation type for which the solutions exhibit strong endpoint and internal boundary-layers. It is clear, of course, that due to the rapid increase in the order of the compound system, the method is not likely to be practical for problems of order higher than six. Nevertheless, our present results for fourth- and sixth-order problems are directly applicable to a large class of problems including many which arise from the study of hydrodynamic stability. Moreover, the basic ideas involved can easily be adapted to deal with problems involving systems of odd order, an example of which can be found in a recent study of the stability of the similarity solutions for swirling flow above an infinite rotating disk [7].

We also wish to note that in the course of our study of the compound matrix method for sixth-order systems, we have derived a number of quadratic identities in addition to those given by (2.35). For example, by systematically replacing with zeros two of the rows in the first three columns of the matrix

$$\left[\begin{array}{ccc|ccc} u_1 & v_1 & w_1 & u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 & u_2 & v_2 & w_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_6 & v_6 & w_6 & u_6 & v_6 & w_6 \end{array}\right] \tag{6.1}$$

and then expanding the corresponding determinants in terms of complementary minors, we obtain fifteen identities[1] each consisting of four terms which are quadratic in the components of $\mathbf{y}$. Moreover, these identities appear to be independent from those given by (2.35). Similarly, by setting in turn to zero one of the first six rows in the first four columns of the matrix

$$\left[\begin{array}{cccc|ccc} g_1 & u_1 & v_1 & w_1 & u_1 & v_1 & w_1 \\ g_2 & u_2 & v_2 & w_2 & u_2 & v_2 & w_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_6 & u_6 & v_6 & w_6 & u_6 & v_6 & w_6 \\ \hline 0 & 0 & 0 & 0 & u_k & v_k & w_k \end{array}\right] \tag{6.2}$$

and letting $k = 1, 2, \ldots, 6$, and then expanding the corresponding determinants, we obtain thirty quadratic identities involving products of the components of $\mathbf{y}$ and $\mathbf{z}$. We have yet, however, to explore the full implications of these further identities since thus far they do not appear to play a central role in the overall development of the compound matrix method. Nevertheless, it would certainly be desirable to study these identities more systematically in the future since they will very likely be crucial to the further clarification of the relation between the compound matrix method and the Riccati method when they are applied to inhomogeneous boundary-value problems with general boundary conditions. One can perhaps even speculate that a systematic application of these identities may lead to certain significant simplifications of the compound matrix method in some special cases.

Acknowledgments

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[1] Dr. A. Davey (1979, unpublished) has shown that only five of these fifteen identities are linearly independent.

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